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The State Ratchet

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The State Ratchet[1]

Consider a general life-history model. Assume there is a single state, $\xi(a)$, that describes the size or strength or vitality of an individual at age a. Let $\pi(a)$ be the proportion of an individual's available resources at age a that is invested in increasing and maintaining $\xi(a)$. The value of $\xi(a)$ influences mortality and/or fertility at age a and/or subsequently. Remaining resources $1 - \pi(a)$ at age a can be used to increase fertility and/or to lower mortality at age a and/or subsequently.

The value of $\xi(a)$ increases if but only if $\pi(a)$ exceeds some index of deterioration $\delta(\xi) \equiv \delta(\xi(a))$. Note that δ is a function of ξ and hence indirectly a function of a. If $\pi(a) > \delta(\xi)$, then $\xi(a)$ increases. If $\pi(a) = \delta(\xi)$, then $\xi(a)$ remains unchanged. And if $\pi(a) < \delta(\xi)$, then $\xi(a)$ decreases. This makes $\pi(a)$ a control variable. The trajectory of $\pi(a)$ is assumed to determine the trajectory of $\xi(a)$ via an autonomous first order differential equation,

$$\dot{\xi} = g(\xi(a), \pi(a)). \tag{1}$$

An individual starts life by developing, so $\pi(0) > \delta(\xi(0))$.

The optimal strategy of allocations $\pi(a)$ over the life-course is the strategy that maximizes Darwinian fitness, measured as lifetime reproductive success, a functional of the form

$$\max R = \int_0^\infty f(\xi(a), \pi(a))\, da \tag{2}$$

where $f(\xi(a), \pi(a))$ depends on the age-trajectories of mortality and fertility and hence on the age trajectories of $\xi(a)$ and $\pi(a)$. By using the following theorem, it is possible to determine the general nature of the optimal strategy.

The State Ratchet: Consider an optimization problem associated with an objective as given in Eq. 2 that is solely determined by a single state that changes continuously over age according to an autonomous first order differential equation as given in Eq. 1. If an optimal solution exists and each state is associated with exactly one optimal strategy, then the state trajectory must be a monotonic function over age. Once the organism chooses to maintain a state for any finite interval, it will maintain this state forever.

The state ratchet has an intuitive explanation. Let $\pi^*(a)$ denote the optimal strategy at age a associated with state $\xi(a)$. Assume this strategy implies an increase in $\xi(a)$ to $\xi(a^+) = \xi(a) + \varepsilon$, $\varepsilon > 0$. If at the higher age a^+ the optimal strategy $\pi^*(a^+)$ would lead to a decrease in ξ, then ξ would shrink continuously. Clearly when it reaches its former state $\xi(a^{++}) = \xi(a)$ at some higher age a^{++}, $\pi^*(a^{++})$ is known to be such that ξ increases again since π is solely determined by state and not age. But the continuity in state is even stronger. To reach the higher state $\xi(a) + \varepsilon$ it must have been optimal to grow at all intermediate states between $\xi(a)$ and $\xi(a) + \varepsilon$ so shrinkage would violate the optimal strategy at $\xi(a) + \varepsilon - \iota$ for any $0 < \iota < \varepsilon$ and $\iota \to 0$ and for any $\varepsilon > 0$ and $\varepsilon \to 0$. Consequently, if the optimal strategy at starting age zero implies that $d\xi(a)/da > 0$ at $a = 0$, then $d\xi(a)/da \geq 0$ for any $a > 0$. Similarly, if the optimal strategy at starting age zero implies that $d\xi(a)/da < 0$ at $a = 0$, then $d\xi(a)/da \leq 0$ for any $a > 0$. Finally, if the optimal strategy at starting age zero implies that $d\xi(a)/da = 0$ at $a = 0$, then $d\xi(a)/da = 0$ for any $a > 0$. More generally, if $d\xi(a)/da = 0$, at any age $\hat{a}$ then $d\xi(a)/da = 0$ for all $a > \hat{a}$.

[1] The original title of this working paper was "Monotonic state trajectories from single-state dynamic optimization models"

It should be emphasized that this holds true only if age does not directly influence the optimal solution: The optimal solution must depend on the value of a unique state variable $\xi(a)$. This value can change with age but it is its value and not age per se that is important.

The state ratchet is known from the theory of dynamic optimization. For an infinite horizon autonomous optimal control problem with a single state variable the optimal state path must be monotone (Kamien & Schwartz (1991, p. 179) and Léonard & Van Long (1992, p. 294)).

This result is very general. Applied to the life-course model described above it implies that an organism, initially increasing in state, can never start decreasing. Suppose that the ability to resist death as well as the reproductive potential of an organism rises with increasing size or strength or vitality. Senescence at some age a occurs if and only if $\xi(a)$ declines at this age. Consequently, the state ratchet implies that senescence is impossible.

The state ratchet precludes a non-monotonic optimal state path for any life history model based on a single state variable. To overcome the ratchet but to keep a model that is essentially based on a single state it is necessary to introduce a switch variable. The switch is a binomial indicator that determines whether the organism is in increase or decrease mode. The switch itself does not affect survival or reproduction, which makes it a pseudo state. If an organism "wants" to jump the maintenance barrier the switch needs to change to decrease mode. In this case the optimality of the strategy is not violated as the smaller state is now associated with a different value of the switch. Depending on whether the switch is triggered once or several times, internally or externally, different state trajectories can emerge. Any repeated trajectories of increase and decrease have to be identical.

References

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